Dynamical model of Ising spins

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A two-dimensional dynamical model of Ising spins is introduced. Since we were not able to define energy in our system, we introduced an object called the disagreement function. This function controls the dynamics—minimizing it locally we decide upon spin flipping. Amazingly, local minimization of the disagreement function can lead to an increase of its global value. We present the phase diagram of the system and show that exactly the same initial conditions can lead the system to one of several, completely different final steady states.

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I. INTRODUCTION

Almost a century ago physicists asked the question if phase transitions could be explained by microscopic theory. To answer this question, in 1920 Wilhelm Lenz proposed a very simple microscopic model of interacting spins. Supervised by Lenz, Ernst Ising in his dissertation (1924) studied the special case of a linear chain of magnetic moments [20] which are only able to take two positions "up" and "down" and which are coupled by interactions between nearest neighbors. He showed that spontaneous magnetization cannot be explained using this model in its one-dimensional version. However, later it turned out that the two-dimensional version of the model (known at present as the Ising model) can explain the critical phase transition. This taught us that very simple local interactions can lead to qualitative changes on the macroscopic scale.

Rapid changes on the macroscopic scale (like phase transitions in physics) can be observed in various complex systems—from biological (e.g., mass extinctions or speciation) to financial (crashes, speculative bubbles) or social (sudden social depression or euphoria). These changes are usually unexpected and no obvious source of such a behavior can be identified. In recent years physicists have started to explain these "outside physics" phenomena [1,2] in terms of microscopic interactions, like they have been doing for physical systems.

Recently we have proposed a simple model [3] to describe how opinions spread in human society. The crucial difference of our model compared to voter or Ising-type models is that information flows outward [4]. In our model each site of a one-dimensional lattice carries an Ising spin. Two neighboring parallel spins—i.e., two neighboring people sharing the same opinion—convince their neighbors of this opinion. If they do not have the same opinion, they bring their neighbors to the opposite position. Our model, named by Stauffer the "Sznajd model," has been modified and applied in sociology [5,6], marketing [7,8], finance [9], and politics [10,11]; see also reviews by Stauffer [4] and Schechter [12]. At the same time the model posed new challenges to statistical physics [14]. Several interesting results have been found recently. Stauffer and de Oliveira [13] showed that the density of never-changed opinions in the Sznajd model decays in time as $1/t^{\theta}$ with $\theta=3/8$ for the one-dimensional chain, which is compatible with Ising model results. However, in higher dimensions the exponent differs from the Ising θ. Slanina and Lavicka [14] solved our model analytically on a complete graph, which is a mean-field-like treatment, and showed the existence of a phase transition, which was earlier found using Monte Carlo simulations [15].

Because the outflow of information seemed to be crucial, we decided to introduce a generalized model which kept our old dynamics (the information flows outward), but introduced a function controlling whether a spin should be flipped or not. The model consists of two components (hence the name TC model) [16]:

(i) *Dynamics*—the information flows outward; i.e., a pair of spins S_i and S_{i+1} is chosen to change their nearest neighbor.

(ii) *Disagreement function*—the change of spins is controlled by a certain function, which is locally minimized.

In the original one-dimensional Sznajd model the $(i-1)$th spin is influenced by its two neighbors and the steady state of the system is degenerated—with equal probability the ferromagnetic or the antiferromagnetic state is reached. The one-dimensional TC model was proposed to generalize the one-dimensional Sznajd model. For a system in which the $(i-1)$th spin interacts with its two neighbors the Hamiltonian can be written in the following form:

$$H = -J_1\sum_i S_{i-1}S_i - J_2\sum_i S_{i-1}S_{i+1}. \quad (1)$$

For $J_1>0$ and $J_2<0$ this is the well-known axial next-nearest-neighbor Ising (ANNNI) model introduced in [17], which allowed the system to display frustration. We have used this Hamiltonian to construct the disagreement function E for the one-dimensional TC model [16]:

$$E_{i-1} = -J_1 S_{i-1}S_i - J_2 S_{i-1}S_{i+1}, \quad (2)$$

for any value of coupling constants J_1 and J_2. This allows us to reproduce steady states from the original Sznajd model

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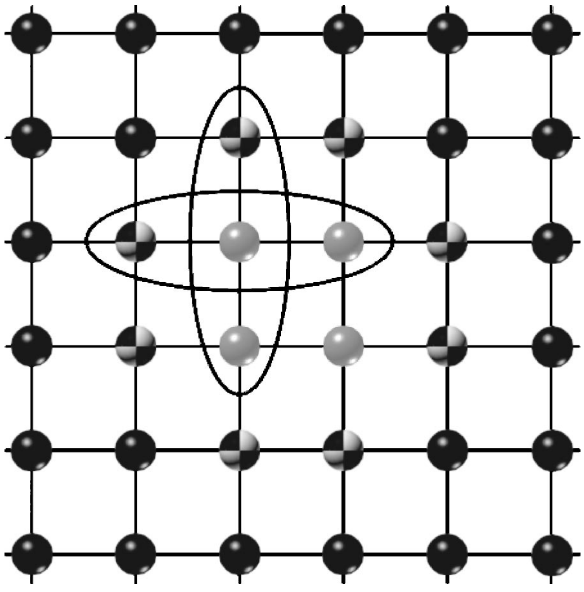

FIG. 1. Transformation of the one-dimensional TC model to two dimensions. The one-dimensional rule is applied to each of the four chains of the 2×2 box (light balls).

(for $|J_1|<|J_2|$). For such values, a double degeneration of the steady state was observed—the ferromagnetic and the antiferromagnetic states were equally probable. It should be noted that this degeneration was obtained even for both coupling constants greater than zero. The full phase diagram of the one-dimensional model consists of four different phases: ferromagnetic, antiferromagnetic, antiphase (2,2), and a double-degenerated one [16]. In this paper we will introduce a two-dimensional version of the TC model. We will show that in this case the number of possible final states increases and that the knowledge of the initial state does not help much in predicting the final state. We will also show how local minimization of the disagreement function influences the global disagreement function.

II. MODEL

We investigate a system of Ising spins on a square lattice. The one-dimensional rule is applied to each of the four chains of four spins each, centered about two horizontal and two vertical pairs of light balls in Fig. 1. The algorithm is the following:

(i) Choose at random a spin—e.g., $S_{i,j}$—which defines a 2×2 box of spins $(S_{i,j},S_{i,j+1},S_{i+1,j},S_{i+1,j+1})$, light balls in Fig. 1.

(ii) Calculate the disagreement function for each of the eight nearest neighbors of the box defined in point (i) (chessboard colored balls in Fig. 1)—e.g., for $S_{i-1,j}$:

$$E_{i-1,j} = -J_1S_{i-1,j}S_{i,j} - J_2S_{i-1,j}S_{i+1,j}\,. \tag{3}$$

(iii) Calculate the disagreement function for each of the eight nearest neighbors of the box in the case of a flipped spin—e.g., for $S_{i-1,j}$:

$$E'_{i-1,j} = J_1S_{i-1,j}S_{i,j} + J_2S_{i-1,j}S_{i+1,j}. \tag{4}$$

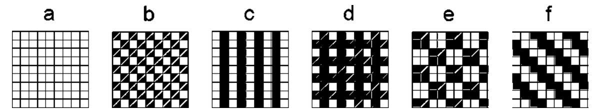

FIG. 2. All possible steady states of the two-dimensional TC model. For example, in the phase *A* we should get steady states consisting of ferromagnetic and antiferromagnetic chains and so there are four possibilities: (a) all chains are ferromagnetic, (b) all chains are antiferromagnetic, (c) there are ferromagnets in columns and antiferromagnets in rows or vice versa, or (d) chains of ferromagnets alternate antiferromagnetic chains. Phase *C* consists of (2,2) antiphase chains, which gives two possibilities in two dimensions (e) or (f).

(iv) For each of the eight spins check the difference in the disagreement function—e.g., for $S_{i-1,j}$: $E'_{i-1,j}-E_{i-1,j}$. If it is smaller than zero, then flip the spin (e.g., $S_{i-1,j}$); otherwise, leave it unchanged.

Recently the following question was raised by Spirin *et al.* [18,19]: What happens when an Ising ferromagnet, with spins endowed with Glauber dynamics, is suddenly cooled from a high temperature to zero temperature? The first expectation was that the system should eventually reach the ground state. However, this is true only for a one-dimensional system. On a square lattice, there exist many metastable states that consist of alternating vertical (or horizontal) stripes of widths $\geqslant 2$. These arise because a straight boundary between up and down phases is stable in zero-temperature Glauber dynamics.

The same question can be asked in respect to the TC model. We start from a completely random system and monitor the evolution of the system.

III. STEADY STATES AND THE PHASE DIAGRAM

In the TC model we change the spin according to its two neighbors—say, $S_{i-1,j}$ according to $S_{i,j}$ and $S_{i+1,j}$. We can easily calculate disagreement function $E_{i-1,j}$, which we denote by E^- for simplicity:

(i) $\uparrow\uparrow\uparrow$, $\downarrow\downarrow\downarrow$, $E_1^-=-(J_1+J_2)$,
(ii) $\uparrow\uparrow\downarrow$, $\downarrow\downarrow\uparrow$, $E_2^-=-J_1+J_2$,
(iii) $\uparrow\downarrow\uparrow$, $\downarrow\uparrow\downarrow$, $E_3^-=J_1-J_2$,
(iv) $\downarrow\uparrow\uparrow$, $\uparrow\downarrow\downarrow$ $E_4^-=J_1+J_2$.

The definition of the model (change of $S_{i-1,j}$ according to $S_{i,j}$ and $S_{i+1,j}$) implies that only two transitions are possible $E_1^-\leftrightarrow E_4^-$ and $E_2^-\leftrightarrow E_3^-$. This defines four phases

(A) $|J_1|<J_2$: $E_1^-<E_4^-, E_3^-<E_2^-$,
(B) $|J_2|<J_1$: $E_1^-<E_4^-, E_2^-<E_3^-$,
(C) $J_2<|J_1|$: $E_4^-<E_1^-, E_2^-<E_3^-$,
(D) $J_1<|J_2|$: $E_4^-<E_1^-, E_3^-<E_2^-$.

This means that, e.g., in phase *A* the ferromagnetic and the antiferromagnetic triplets are preferable—i.e., $\downarrow\uparrow\uparrow\rightarrow\uparrow\uparrow\uparrow$, $\uparrow\uparrow\downarrow\rightarrow\downarrow\uparrow\downarrow$. Thus in one dimension we expect a double degeneration of the steady state: the ferromagnetic and the antiferromagnetic steady states should be possible. Indeed, Monte Carlo simulations confirm this expectation.

In the two-dimensional case, the one-dimensional rule is applied to each of the four chains of the 2×2 box. For this reason the two-dimensional steady states are a combination

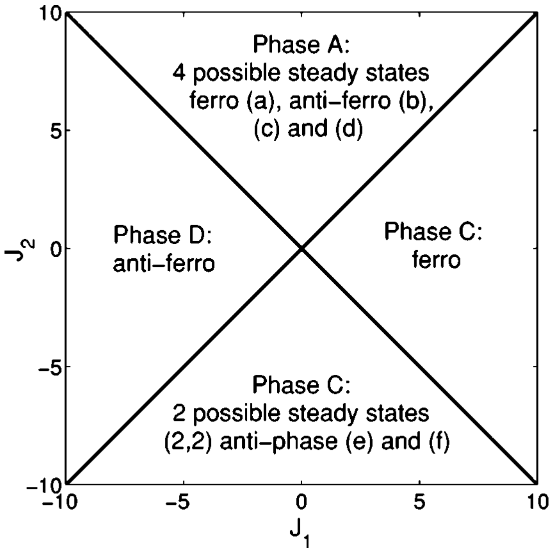

FIG. 3. Phase diagram for the two-dimensional TC model.

of one-dimensional steady states. All possible steady states are presented in Fig. 2 and complete the phase diagram in Fig. 3. The same results can be also obtained from Monte Carlo simulations.

The most interesting is, of course, phase *A* in which four qualitatively different phases are possible. Moreover, knowledge of the initial state does not allow us to predict the final state of the system. In the next section we will show what the probability is of each final state in the case of a random initial state.

IV. PREDICTING THE FINAL STATE OF THE SYSTEM

An investigation of the two-dimensional ferromagnet under Glauber dynamics showed that in zero temperature many final states exist and they all consist of alternating vertical or horizontal stripes [18,19]. This arises because in zero-temperature Glauber dynamics a reversal of any spin along the boundary raises the energy. In the TC model the disagreement function together with new local dynamics leads the system to one of several, structurally different steady states (see Fig. 2). The interesting point is that even if we always start from exactly the same random initial state, we can reach all possible steady states of a given phase. But can we predict the final state of the system? One could guess that the probability of reaching a certain steady state depends on the disagreement function of the state, because the disagreement function plays the role of energy in the TC model. Let us first define the global disagreement function as

$$E = \frac{1}{N}\sum_{i,j} E_{i,j}, \qquad (5)$$

where N is the number of spins in the system and $E_{i,j}$ is the local disagreement function calculated from Eq. (3). The global disagreement function can be calculated easily for each possible steady state. One could expect that the most probable state has the lowest value of E. We have made Monte Carlo (MC) simulations for random initial conditions and have found that this is not true; see Table I.

TABLE I. The global disagreement function (calculated analytically) and probabilities of reaching the steady state (from Monte Carlo simulations) in the phase *A*.

Type	E	$E(J_1=1,J_2=2)$	Probability
a	$-2(J_1+J_2)$	−6	1/8
b	$2(J_1-J_2)$	−2	1/8
c	$-2J_2$	−4	1/4
d	$-2J_2$	−4	1/2

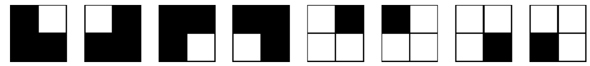

FIG. 4. All equivalent configurations of a type-*d* steady state.

This result shows that in the TC model the probability of the steady state does not depend on its global disagreement function, but only on the number of equivalent configurations connected with each type of the steady state. For example, there are only two configurations for the ferromagnet—all spins up or all spins down, but four different configurations for type *c* and eight configurations for type *d* (see Fig. 4).

The results presented in Table I are quite surprising, because each flip of spin is connected with local minimization of the disagreement function. Thus, it is very interesting to look at the time evolution of the global disagreement function. In Fig. 5 we present a sample plot for $J_1=1$ and $J_2=2$ (phase *A*). The time evolution was obtained from Monte Carlo simulations for a 100×100 square lattice. Averaging was done over 10^4 samples. Initially all evolutions of E are the same. Suddenly, after several hundred Monte Carlo steps, the system "decides" to which steady state it will evolve.

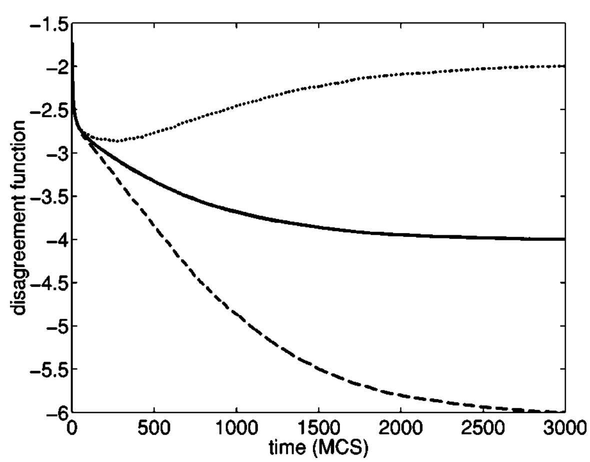

FIG. 5. Time evolution of the global disagreement function in phase *A*. The dotted line presents the evolution to the antiferromagnetic state, the solid line to the both type-*c* and -*d* steady states, and the dashed line to the ferromagnetic state. Initially all evolutions are the same then, after several hundred Monte Carlo steps the system "decides" on one steady state.

Surprisingly, even exactly the same initial states can evolve to different steady states. In the case of the antiferromagnetic steady state, the global disagreement function initially decreases as one would expect. However, after some time it starts to increase, which is quite astonishing.

V. SUMMARY

We proposed a two-dimensional version of the TC model. Simulations and analytical reasoning showed that the TC model depending on two interaction coefficients (J_1, J_2) can eventually lead the system to one of four phases: *A* (degenerated, in which four qualitatively different steady states exist), *B* (ferromagnetic), *C* (double degeneration), and *D* (antiferromagnetic). It should be also noted that in our model a degeneration of the steady state is possible even if both coupling constants are greater or smaller than zero, while in the ANNNI model coupling constants need to have opposite signs in order to obtain degeneration.

Interestingly, the same initial state can evolve to several different limiting structures and we cannot predict which final state will be reached. In the TC model the probability of reaching a final state does not depend on its disagreement function, but only on the number of possible representations of the state. Thus, for example, in phase *A* the ferromagnetic state has the lowest disagreement function, yet it is highly improbable; see Table I. Although the disagreement function is minimized locally, the global disagreement function—defined as a sum of the local disagreement functions—can increase during the evolution.

Finally, the TC model is an example of what can happen when we introduce a function which has only local meaning and try to predict what happens to the whole system. We have to leave aside all our physical intuitions, although the model looks very "physical" at first sight.

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